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On the number of clique Boolean functions

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Abstract

An explicit formula for the number of n-variable "clique functions" is given, which contain "bad" parameters related to the numbers of certain monotone functions. We compute the number of n-variable clique functions for up to $n = 7$ through the evaluation of the parameters.

List of Notations

E	$\{0,1\}$
E^n	n-dimensional cube
P	the set of Boolean functions
M	the set of monotone increasing functions
N	the set of clique functions
$\boldsymbol{a}$	a point in E^n, i.e. a vector $a_1 \cdots a_n$, where $a_i \in E$, $1 \leq i \leq n$
$\mathbf{0}, \mathbf{1}$	$0 \cdots 0$, $1 \cdots 1$
$\boldsymbol{a}$ and $\boldsymbol{b}$ intersecting	$\boldsymbol{a} \wedge \boldsymbol{b} \neq \mathbf{0}$
$\boldsymbol{a} \preceq \boldsymbol{b}$	$a_1 \leq b_1, \ldots, a_n \leq b_n$
E^n_-, E^n_+	lower and upper halves of the cube
I_f	$= \{\boldsymbol{a} \mid f(\boldsymbol{a}) = 1\}$
I_f^-	$= \{\boldsymbol{a} \mid f(\boldsymbol{a}) = 1$ and $f(\boldsymbol{b}) = 1$ for no $\boldsymbol{b} \preceq \boldsymbol{a}, \boldsymbol{b} \neq \boldsymbol{a}\}$
O_f	$= \{\boldsymbol{a} \mid f(\boldsymbol{a}) = 0\}$
O_f^+	$= \{\boldsymbol{a} \mid f(\boldsymbol{a}) = 0$ and $f(\boldsymbol{b}) = 0$ for no $\boldsymbol{b} \succeq \boldsymbol{a}, \boldsymbol{b} \neq \boldsymbol{a}\}$
$Q(n)$	$= \{f \mid f \in M \cap N,\ I_f^- \subseteq E^n_-\}$
$Q_r(n)$	$= \{f \mid f \in Q(n)$ and $\lvert I_f^- \rvert = r\}$
$\overline{\boldsymbol{a}}$	complement of $\boldsymbol{a}$, i.e. $\overline{\boldsymbol{a}} = \bar{a}_1 \cdots \bar{a}_n$
$\overline{A}$	$= \{\overline{\boldsymbol{a}} \mid \boldsymbol{a} \in A\}$
$\lfloor a \rfloor$	the largest integer $\leq a$, i.e floor of a

1. Introduction

Let $E = \{0, 1\}$. A Boolean functions $f : E^n \to E$ is called a *clique function* if it satisfies the following condition (1):

$$\text{if } f(x_1, ..., x_n) = f(y_1, ..., y_n) = 1 \text{ then } x_i = y_i = 1 \text{ for some } i. \tag{1}$$

If $a_i = b_i = 1$ then the vectors $\boldsymbol{a} = a_1 \cdots a_n$ and $\boldsymbol{b} = b_1 \cdots b_n$ are said to be *intersecting* at the i-th coordinate. By taking as vertices the vectors $\boldsymbol{x}$ such that $f(\boldsymbol{x}) = 1$, and connecting every pair of intersecting vectors by an edge, we obtain a "complete graph" or a "clique" in the graph theoretical terminology, under the above-mentioned condition (1). This is why we use the word "clique" for such a function.

The set of clique functions has been investigated in several papers: it is known to be a submaximal set of functions in the ordinary Post algebra [Pos21], and it is a maximal set in some modified Post algebra (cf. [Ibu68,KaF78,Noz82,MSHMF88]). It appears also in the universal algebra where the functional constructions are studied from algebraic standpoint [Ber83].

We are interested in counting the number of clique functions with n variables, since it has been paid no attention so far. It is deeply related to the number of monotone functions with n variables (the famous Dedekind's problem [Ded97]), on which there are a number of investigations [Kle69,Kor81,Hro85].

In Section 3 we give a formula for the number of n-variable clique functions. Although the formula is an explicit one, it contains a series of "bad" parameters related to the number of certain monotone functions. In Section 4 we give an efficient algorithm to evaluate the parameters, and determine the numbers of clique functions for up to $n = 7$.

2. Definitions and Notations

The set of n-variable clique functions is denoted by $N(n)$. The set of n-variable monotone (increasing) functions is denoted by $M(n)$, which is defined by

$M(n) = \{f | f(x_1, ..., x_n) \leq f(y_1, ..., y_n) \text{ if } x_i \leq y_i \text{ for all } i\}$.

Let c_0 and c_1 be the constant-valued functions of n-variables assuming the values 0 and 1, respectively (we allow to write c_0 and c_1 regardless of the number of its variables). For a set F we denote the number of its elements by $|F|$.

Let $\boldsymbol{a} = a_1 \cdots a_n$, $\boldsymbol{b} = b_1 \cdots b_n \in E^n$. We denote $\boldsymbol{a} \preceq \boldsymbol{b}$ if $a_i \leq b_i$ for all i, $1 \leq i \leq n$. For each vector $\boldsymbol{a}$ we define *lower shadow* $S^-(\boldsymbol{a}) := \{\boldsymbol{b} | \boldsymbol{b} \preceq \boldsymbol{a}\}$ and *upper shadow* $S^+(\boldsymbol{a}) := \{\boldsymbol{b} | \boldsymbol{a} \preceq \boldsymbol{b}\}$. Finally we set $S^+(A) := \cup_{\boldsymbol{a} \in A} S^+(\boldsymbol{a})$ and $S^-(A) := \cup_{\boldsymbol{a} \in A} S^-(\boldsymbol{a})$.

For a function f, a 0-point (or 1-point) of f is a vector $\boldsymbol{x}$ such that $f(\boldsymbol{x}) = 0$ (or $f(\boldsymbol{x}) = 1$, respectively). Let us denote the set of *1-points* and the set of *0-points* of f by $I_f := \{\boldsymbol{a} | f(\boldsymbol{a}) = 1\}$ and $O_f := \{\boldsymbol{a} | f(\boldsymbol{a}) = 0\}$, respectively. Now define two sets of extremum points: the set of *maximum 0-points*, denoted by O_f^+ (which is the set $\{\boldsymbol{a} | f(\boldsymbol{a}) = 0 \text{ and } f(\boldsymbol{b}) = 1 \text{ for all } \boldsymbol{b} \in S^+(\boldsymbol{a}) \setminus \{\boldsymbol{a}\}\}$) and the set of *minimum 1-points*, denoted by I_f^- (which is the set $\{\boldsymbol{a} | f(\boldsymbol{a}) = 1\ f(\boldsymbol{b}) = 0 \text{ for all } \boldsymbol{b} \in S^-(\boldsymbol{a}) \setminus \{\boldsymbol{a}\}\}$). The set of minimum 1-points for the function c_0 and the set of maximum 0-points for the function c_1 are defined to be empty.

Let $A = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k\}$ be a set of k points. We call A *incomparable* if each two points $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$ are incomparable, i.e. $\boldsymbol{a}_i \not\preceq \boldsymbol{a}_j$ for any distinct suffices i and j. A singleton set $\{\boldsymbol{a}\}$ is incomparable. The set A is called *intersecting* if every pair of elements are intersecting. A singleton set $\{\boldsymbol{a}\}$ is intersecting except when $\boldsymbol{a} = \mathbf{o}$. Thus every singleton set is incomparable and intersecting except when $\boldsymbol{a} = \mathbf{o}$.

Lemma 2.1. *Let $A = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k\}$ be a set of m points and $O_f^+ = A$ (or $I_f^- = A$) for a function $f \in P$. Then A is incomparable. Conversely, for any incomparable set A there exists a function f such that $O_f^+ = A$ and a function f' such that $I_{f'}^- = A$.*

Proof. Obvious. □

Corollary 2.1. *For any incomparable set A there exists a unique monotone function $f \in M$ such that $O_f^+ = A$ and a unique monotone function $f' \in M$ such that $I_{f'}^- = A$.*

3. The number of n-variable clique functions

Let E_t^n be the t-th *layer* of the cube E^n, i.e. $E_t^n = \{\boldsymbol{a} \in E^n | w(\boldsymbol{a}) = t\}$, where $w(\boldsymbol{a})$ denote the number of 1's in $\boldsymbol{a}$. For n even let $E^n_{n/2:upper}$ denote the *upper half of the mid-layer*, i.e. $\{\boldsymbol{a} | \boldsymbol{a} = 1a_2 \ldots a_n \text{ and } w(\boldsymbol{a}) = n/2\}$. Let E_-^n denote the *lower half of the cube* which is defined by $E_-^n = \{\boldsymbol{a} \in \cup_{t=0}^{[n/2]} E_t^n \setminus E^n_{n/2:upper}\}$. The *upper half of the cube* E_+^n is also defined by $E_+^n = E^n \setminus E_-^n$.

Lemma 3.1. *Let f be a Boolean function. If $I_f \subseteq E_+^n$, then $f \in N$.*

Proof. Consider $\boldsymbol{a} = a_1 \cdots a_n$ and $\boldsymbol{b} = b_1 \cdots b_n$ such that $f(\boldsymbol{a}) = f(\boldsymbol{b}) = 1$. Then $\boldsymbol{a}, \boldsymbol{b} \in I_f \subseteq E^n_+$. Hence $w(\boldsymbol{a}) \geq n/2$ and $w(\boldsymbol{b}) \geq n/2$. If $w(\boldsymbol{a}) > n/2$ or $w(\boldsymbol{b}) > n/2$ then $\boldsymbol{a} \wedge \boldsymbol{b} \neq \mathbf{o}$ is obvious. In the case $w(\boldsymbol{a}) = w(\boldsymbol{b}) = n/2$ we have $\boldsymbol{a} \wedge \boldsymbol{b} = 1c_2 \ldots c_n \neq \mathbf{o}$ by definition of E^n_+. □

Lemma 3.2. *If $\boldsymbol{a} \wedge \boldsymbol{b} \neq \mathbf{o}$ then $\boldsymbol{a}' \wedge \boldsymbol{b}' \neq \mathbf{o}$ for any $\boldsymbol{a} \in S^+(\boldsymbol{a})$ and any $\boldsymbol{b} \in S^+(\boldsymbol{b})$.*

Proof. Obvious. □

Lemma 3.3. *If $\boldsymbol{a} \wedge \boldsymbol{b} = \mathbf{o}$ then $\boldsymbol{a} \preceq \overline{\boldsymbol{b}}$ and $\boldsymbol{b} \preceq \overline{\boldsymbol{a}}$.*

Proof. Obvious. □

In the sequel we need special subsets of monotone clique functions defined as follows.

$Q(n) := \{f | f \in M \cap N \text{ and } I^-_f \subseteq E^n_-\}$ and $Q_r(n) := \{f \in Q(n) | |I^-_f| = r\}$.

Note that $Q(n) = Q_0(n) \cup Q_1(n) \cup \cdots Q_m(n)$, where m is the maximum number of the elements of the sets of minimum 1-points for all f satisfying $I^-_f \subseteq E^n_-$. We will determine m later.

Consider $f \in Q_r(n)$. Define *extended shadow of f* by

$$X_f := \{S^+(I^-_f) \setminus I^-_f\} \cup \{E^n_+ \setminus S^-(\overline{I^-_f})\},$$

where $\overline{I^-_f}$ is the complement of I^-_f, i.e. $\overline{I^-_f} = \{\overline{a} | a \in I^-_f\}$.

Lemma 3.4. *For any $f \in Q(n)$ and any $A \subseteq X_f$, if $A \cup I^-_f = I_g$ then $g \in N$.*

Proof. We must show $\boldsymbol{a}' \wedge \boldsymbol{b}' \neq \mathbf{o}$ for any $\boldsymbol{a}, \boldsymbol{b} \in A \cup I^-_f$. There are three possibilities.

1) $\boldsymbol{a}, \boldsymbol{b} \in E^n_+$ then $\boldsymbol{a}' \wedge \boldsymbol{b}' \neq \mathbf{o}$ by Lemma 3.1.

2) $\boldsymbol{a}, \boldsymbol{b} \in E^n_-$ then $\boldsymbol{a}' \wedge \boldsymbol{b}' \neq \mathbf{o}$ by Lemma 3.2.

3) $\boldsymbol{a} \in E^n_-$, $\boldsymbol{b} \in E^n_+$. Assume $\boldsymbol{a} \wedge \boldsymbol{b} = \mathbf{o}$. Then by Lemma 3.3

$$\boldsymbol{b} \preceq \bar{\boldsymbol{a}}. \tag{2}$$

Since $\boldsymbol{a} \in E^n_-$ there is $\boldsymbol{c} \in I^-_f$ such that $\boldsymbol{c} \preceq \boldsymbol{a}$. Therefore

$$\bar{\boldsymbol{a}} \preceq \bar{\boldsymbol{c}}. \tag{3}$$

From (2) and (3) follows

$$\boldsymbol{b} \preceq \bar{\boldsymbol{c}}.$$

Thus $b \in S^-(\overline{I_f^-})$, hence $b \notin X_f$. This means $b \notin A$. Since $b \notin I_f^-$ (because $b \in E_+^n$), we have $b \notin A \cup I_f^-$. A contradiction. □

Lemma 3.5. *For $f \in Q_r(n)$ set $D(f) := \{g \in N | I_g^- \cap E_-^n = I_f^-\}$. Then*

$$|D(f)| = 2^{2^{n-1}-r}.$$

Proof. Consider extended shadow X_f. By Lemma 3.4 $D(f)$ is the set of all functions constructed by choosing arbitrary values (0 or 1) for $a \in X_f$ and setting value 1 for any $a \in I_f^-$. From this follows $|D(f)| = 2^{|X_f|}$. Since $X_f = \{S^+(I_f^-) \setminus I_f^-\} \cup \{E_+^n \setminus S^-(\overline{I_f^-})\}$ and $S^+(I_f^-) \cap S^-(\overline{I_f^-}) = \phi$, in view of the symmetry we have $|S^+(I_f^-)| = |S^-(\overline{I_f^-})|$, and finally $|X_f| = |E_+^n| - |I_f^-| = 2^{n-1} - r$. □

We are going to count the number of n-ary clique functions in a systematic way by partitioning them into the following $m+1$ classes according to the size of $I_f^- \cap E_-^n$: class r is the set of clique functions f such that

$$|I_f^- \cap E_-^n| = r.$$

Thus we have the following equality.

Lemma 3.6.

$$|N(n)| = 2^{2^{n-1}} + 2^{2^{n-1}-m} \sum_{r=1}^{m} |Q_r(n)| 2^{m-r},$$

where $m := max_{f \in Q_r(n)} |I_f^-|$.

Proof. 1) Case $|I_f^- \cap E_-^n| = 0$. The number of clique functions is $2^{2^{n-1}}$ by Lemma 3.1. 2) Case $|I_f^- \cap E_-^n| = i$. For each function $g \in Q_r(n)$ we have $2^{2^{n-1}-r}$ clique functions f which satisfies $|I_f^- \cap E_-^n| = r$ and $I_f^- \cap E_-^n = I_g^-$. Hence the number of clique functions satisfying $|I_f^- \cap E_-^n| = r$ is $2^{2^{n-1}-m} \cdot |Q_r(n)| 2^{2^{n-1}-r}$. Therefore we have $|N(n)| = 2^{2^{n-1}} + |Q_1(n)| 2^{2^{n-1}-1} + \cdots + |Q_r(n)| 2^{2^{n-1}-m}$. □

Now we determine the number m.

We introduce a notation. For a subset $A \subseteq E_t^n$ of t-th layer we denote the subsets of its shadow that are included in $t+1$-th and $t-1$-th layer, respectively, by

$$S^{+1}(A) := E_{t+1}^n \cap S^+(A),$$

$$S^{-1}(A) := E_{t-1}^n \cap S^-(A).$$

We have the following lemma.

Lemma 3.7. *For $A \subseteq E_t^n$, $1 < t < \lfloor n/2 \rfloor$ holds*

$$|S^{+1}(A)| > |A|.$$

Proof. cf. [Kor81, Lemma 1.1, p.9]. Note that we have strict inequality in above case. □

Lemma 3.8. *For $f \in Q(n)$, if $I_f^- \not\subseteq E_{\lfloor n/2 \rfloor}^n$ for n odd or $I_f^- \not\subseteq E_{n/2:lower}^n$ otherwise, then there is $g \in Q(n)$ such that $|I_g^-| > |I_f^-|$ and $I_g^- \subseteq \begin{cases} E_{\lfloor n/2 \rfloor}^n & n \text{ odd} \\ E_{n/2:lower}^n & \text{otherwise} \end{cases}$.*

Proof. We consider two cases n odd and n even separately.

1) n odd. Let t_1 be the lowest layer which contain at least one minimum 1-point of f, i.e.

$$E_{t_1}^n \cap I_f^- \neq \phi \text{ and } E_t^n \cap I_f^- = \phi \text{ for } t < t_1.$$

If $t_1 = \lfloor n/2 \rfloor$ then we are done. Now $t_1 < \lfloor n/2 \rfloor$. Consider $A := S^{+1}(E_{t_1}^n \cap I_f^-)$. The following properties hold for A.

1. $I_f^- \cap A \neq \phi$ (by Lemma 3.2).
2. $\boldsymbol{a}$ and $\boldsymbol{b}$ are intersecting for any $\boldsymbol{a}, \boldsymbol{b} \in A$ (by Lemma 3.1).
3. For any $\boldsymbol{a}, \boldsymbol{b}$ such that $\boldsymbol{a} \in A$ and $\boldsymbol{b} \in I_f^- \setminus E_{t_1}^n$ $\boldsymbol{a}$ and $\boldsymbol{b}$ are intersecting because for $\boldsymbol{a}$ there is $\boldsymbol{a}' \in I_f^- \cap E_{t_1}^n$ such that $\boldsymbol{a}' \preceq \boldsymbol{a}$ and $\boldsymbol{a}' \wedge \boldsymbol{b} \neq \mathbf{o}$. Hence we have $\boldsymbol{a} \wedge \boldsymbol{b} \neq \mathbf{o}$ by Lemma 3.1. Therefore the set $(I_f^- \setminus (E_{t_1}^n \cap I_f^-)) \cup A$ is intersecting and incomparable. Hence there is $f_1 \in Q(n)$ such that $I_{f_1}^- = (I_f^- \setminus E_{t_1}^n) \cup A$. By Lemma 3.7

$$|A| = |S^{+1}(E_{t_1}^n \cap I_f^-)| < |E_{t_1}^n \cap I_f^-|.$$

Therefore

$$|I_{f_1}^-| > |I_f^-|.$$

This procedure of constructing f_1 from f (lifting of minimum 1-points) can be repeated until the whole set of minimum 1-points goes into $E_{\lfloor n/2 \rfloor}^n$. This last function is the function g we need.

2) n even. The above procedure of lifting minimum 1-points still works up to $E_{n/2-1}^n$ in this case, too. However, we have just one half of $E_{n/2}^n$. So lifting to $E_{n/2}$ directly doesn't work. Now consider a function f' such that

$$I_{f'}^- = A \cup B, \ A \subseteq E_{n/2-1}^n \text{ and } B \subseteq E_{n/2:lower}^n.$$

It is easy to see that $B \cap S^{+1}(A) = \phi$. Let $S^{+1}(A) = V_+ \cup V^-$, where

$$V_+ \subseteq E^n_{n/2:upper},\ V_- \subseteq E^n_{n/2:lower}$$

The following properties hold.

$$\begin{array}{rrcl} 1. & \overline{V_+} & \subseteq & E^n_{n/2:lower}, \\ 2. & V_- \cap B & = & \phi, \\ 3. & \overline{V_+} \cap B & = & \phi, \\ 4. & \overline{V_+} \cap V_- & = & \phi. \end{array}$$

The properties 1 and 2 are obvious.

3. Assume that there is a vector $\boldsymbol{a}$ in $\overline{V_+} \cap B$. We have $\boldsymbol{a} \in B$ and $\overline{\boldsymbol{a}} \in V_+$. There is $\boldsymbol{a}' \in A$ such that $\boldsymbol{a}' \preceq \overline{\boldsymbol{a}}$. Since for $\boldsymbol{a} \in B$ and $\boldsymbol{a}' \in A$ we have $\boldsymbol{a} \wedge \boldsymbol{a}' = \mathbf{o}$: a contradiction.

4. Assume that there exists $\boldsymbol{a} \in \overline{V_+} \cap V_-$. We have $\boldsymbol{a} \in V_-$ and $\overline{\boldsymbol{a}} \in V_+$. There are $\boldsymbol{a}', \boldsymbol{a}'' \in A$ such that $\boldsymbol{a}' \preceq \boldsymbol{a}$ and $\boldsymbol{a}'' \preceq \overline{\boldsymbol{a}}$. Since $\boldsymbol{a} \wedge \overline{\boldsymbol{a}} = \mathbf{o}$, we have $\boldsymbol{a}' \wedge \boldsymbol{a}'' = \mathbf{o}$. A contradiction.

Now using properties 1–4 and Lemma 3.7, we obtain

$$|I^-_{f'}| = |A| + |B| < |S^{+1}(A)| + |B| = |V_+| + |V_-| + |B| = |\overline{V_+}| + |V_-| + |B|$$
$$\leq |E^n_{n/2:lower}|.$$

Therefore for the function $g \in Q(n)$ defined by $I^-_g := \overline{V_+} \cup V_- \cup B$ (it is easy to check that $g \in Q(n)$) we have $|I^-_g| > |I^-_{f'}|$. □

Here we need a lemma from a result in [Kat68] which is a special case of an Erdoš-problem concerning a set of finite sets.

Lemma 3.9. *Assume that n is even and $C \subset E^n_{n/2}$. If $|C| \leq \frac{1}{2}\binom{n}{n/2}$ then $|S^{-1}(C)| \geq |C|$ where the equality holds only when $|C| = \frac{1}{2}\binom{n-1}{(n-1)/2}$.*

Proof. Our lemma is a direct consequence of [Kat68, Lemma 7, p. 205, eq. (67)]. □

We are to determine m the maximal possible number of minimum 1-points for f where f runs through $Q(n)$. From Lemma 3.8 the number of elements of a maximum intersecting set in the top layer of E^n_- gives it. This is determined by the following theorem.

Theorem 3.1. *The number of elements of a maximum intersecting set in E^n_- is*

$$m = \frac{\lfloor n/2 \rfloor}{n} \cdot \binom{n}{\lfloor n/2 \rfloor}.$$

Proof. We separate two cases. 1) n even. We are to find a maximal intersecting subset of the top layer of E^n_-: $E_{n/2:lower}$. It is easy to see that $E_{n/2:lower}$ itself is such a set. Indeed, for each $\boldsymbol{a} \in E_{n/2:lower}$ we have $w(\boldsymbol{a}) = n/2$ and $\boldsymbol{a}$ begins with a leading 0, i.e. $\boldsymbol{a} = 0a_2 \ldots a_n$. Hence each pair in $E_{n/2:lower}$ is intersecting. Thus $m = \frac{1}{2}\binom{n}{n/2}$.

2) n odd. We show that the set of all points in $E^n_{(n-1)/2}$ having a common intersecting coordinate i is a maximal intersecting set, where i may be any of $1 \leq i \leq n$. To show this assume that a maximal intersecting set C is divided into two subsets by the first bit, i.e. $C = 1A + 0B$, where $A := \{\boldsymbol{a} | \boldsymbol{a} = a_2 \cdots a_n \text{ and } 1a_2 \cdots a_n \in C\}$ and $B := \{\boldsymbol{b} | \boldsymbol{b} = b_2 \cdots b_n \text{ and } 0b_2 \cdots b_n \in C\}$. Thus A and B are subsets of E^{n-1} and B is intersecting. Further, each pair of $n-1$-vectors, one from A and the other from B, is intersecting (*). Now the sets $0\overline{A}$ and $1\overline{B}$ are subsets of $E^n_{(n+1)/2}$. Further, $S^{-1}(1\overline{B}) = 0\overline{B} + 1S^{-1}(\overline{B})$ and obviously $0\overline{B}$ and $1S^{-1}(\overline{B})$ are subsets of $E^n_{(n-1)/2}$. We separate two cases.

2.1) $A = E^{n-1}$. We show that $\boldsymbol{a} \notin A$ for each $\boldsymbol{a} \in S^{-1}(\overline{B})$. Assume $\boldsymbol{a} \in A$ for some $\boldsymbol{a} \in S^{-1}(\overline{B})$. Then there is $\boldsymbol{a'} \in \overline{B}$ such that $\boldsymbol{a'} \succeq \boldsymbol{a}$. That is, $\overline{\boldsymbol{a'}} \in B$, and this means $\boldsymbol{a} \wedge \overline{\boldsymbol{a'}} \neq \mathbf{o}$ from the above-mentioned property (*), but this is a contradiction. Since A contains all $n-1$-vectors, B should be empty.

2.2) $A \neq E^{n-1}$. We may assume that B is not empty. We have

$$|B| \leq \frac{1}{2}\binom{n-1}{(n-1)/2}.$$

Because, otherwise we have some $\boldsymbol{b} \in E^{n-1}_{(n-1)/2}$ such that $\boldsymbol{b}, \overline{\boldsymbol{b}} \in B$ which contradicts that B is an intersecting set. Then from Lemma 3.9 we have $|S^{-1}(\overline{B})| \geq |B|$ (note that $|B| = |\overline{B}|$). Then obviously $D := 1A + 1S^{-1}(\overline{B})$ is intersecting and $|D| \geq |C|$. From maximality of C we have $|C| = |D|$, that is D is also maximal. Then D must contain all vectors with the leading bit 1 (otherwise D can be extended to such a set, contradicting maximality of D), i.e. $D = \{1\boldsymbol{a} | \boldsymbol{a} \in E^{n-1},\ w(\boldsymbol{a}) = (n-3)/2\}$. Hence we have $m = \binom{n-1}{(n-3)/2}$.

Combining the results of cases 1) and 2) we obtain the formula for m, which is indicated in the theorem. □

Let $k(r,n) := |Q_r(n)|$, i.e. $k(r,n)$ is the number of sets $\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_r|\ \boldsymbol{a}_i$ and $\boldsymbol{a}_j$ are intersecting and incomparable in $E^n_-\}$. By definition we set $k(0,n) := 1$. From Lemma 3.6 and Theorem 3.1 we have

Theorem 3.2.

$$|N(n)| = 2^{2^{n-1}-m} \cdot \sum_{r=0}^{m} 2^{m-r} \cdot k(r,n),$$

where $m = \frac{\lfloor n/2 \rfloor}{n} \cdot \left(\begin{array}{c} n \\ \lfloor n/2 \rfloor \end{array} \right)$.

Note that the values $k(r,n)$ can be known for some r through combinatorial considerations: $k(0,n) := 1, k(1,n) = 2^{n-1}-1, k(2,n) = (1/2)\sum_{t=1}^{\lfloor (n-1)/2 \rfloor} \left(\begin{array}{c} n \\ t \end{array} \right) (2^{t-1}-1)(2^{n-t}-2)+(1+(-1)^n)/2 \left(\begin{array}{c} n \\ n/2 \end{array} \right) (2^{n/2-1}-1)(2^{n/2}-2)$ and $k(m,n) = 1$ for n even and $= n$ for n odd.

4. Algorithm for enumerating $Q_r(n)$

We are to count the numbers of elements of $Q_r(n)$ for $r = 1,\ldots,m$. We conveniently represent the cube E^{n-1} by the set of 2^{n-1} integers $\{0,\ldots,2^{n-1}-1\}$ represented by usual binary number system, where each integer $0 \leq k \leq 2^{n-1}-1$ corresponds to a vector in E^{n-1}. We generate each r-subset (subsets containing r elements) of the set $\{1,2,\ldots,2^{n-1}-1\}$ for $1 \leq r \leq m$ and check whether it is incomparable and intersecting (we delete the point $\mathbf{o} = 0\cdots0$ from our consideration since no subset is intersecting if it contains $\mathbf{o}$). In view of the following "saturation" property which both incomparability and intersection obey, lexicographic enumeration of all subsets of $\{1,...,2^{n-1}\}$ is efficient since we can use "cut" of enumeration (cf. [StM88]):

if $\{a_1,\ldots,a_{r-1}\}$ is not intersecting (incomparable), then
$\{a_1,\ldots,a_{r-1},a_r\}$ is not intersecting (incomparable).

In the sequel, we assume that each r-subset is represented as a sequence $a_1a_2\ldots a_r$ where $1 \leq a_1 < \ldots < a_r \leq 2^{n-1}$. Recall the definition of lexicographic order of subsets. For two subsets $a = (a_1,\ldots,a_p)$ and $b = (b_1,\ldots,b_q)$, $a < b$ is satisfied if and only if there exists i $(1 \leq i \leq q)$ such that $a_j = b_j$ for $1 \leq j < i$ and either $a_i < b_i$ or $p = i-1$. This order has an important property that enables efficient enumeration of all subsets having the above-mentioned property.

The lexicographic enumeration of subsets (cf. [NiW78]) goes in the following manner (for example, let $n = 3$):

1, 12, 123,
  13,
2, 23,
3.

Determination of the next subset is executed in two phases. The enumeration is in "extend" phase when it goes from "left" to "right" staying in a row. If the last element of a subset reaches $2^{n-1}-1$ then the algorithm shifts to the next row. We call this phase "reduce" phase. Besides these two phases we will use in the algorithm below another phase called "cut" phase. This phase will be used when the algorithm goes from some subset to another subset in a lower row (not necessarily in the subsequent row), skipping several subsets.

The "cut" occurs in our algorithm either when r-subset is not incomparable or not intersecting or when the number r of the elements in a subset is greater than m. The last case can be conveniently implemented in the extend phase, since r increases only in this phase. It is easy to see that each subset containing 2^i for $i = 0, \ldots, n-1$ is comparable if it is intersecting, hence we can skip these subsets. However, it is not efficient to insert a check of 2^i in the algorithm. It is sufficient to start our enumeration from 2 skipping $2^0 = 1$.

In PASCAL-like notation we present the algorithm for enumerating all r-subsets of E^{n-1} $(1 \leq r \leq m)$ that are incomparable and intersecting (cf. [StM88]). Every r-subset of $\{2, \ldots, 2^{n-1}-1\}$ is represented in the algorithm below by a sequence $j_1, \ldots, j_r$, $1 \leq r \leq m$, $2 \leq j_1 < \ldots < j_r \leq 2^{n-1}-1$.

Note that the singleton set $\{1\}$ and $\{2^{n-1}-1\}$ is never checked in this algorithm, so the obtained $k(1,n)$ should be increased by 2 after the algorithm. The incomparability and intersection of $j_1, \ldots, j_r$ can be checked easily (it requires at most r incomparability and intersection checks assuming inductively that $j_1, \ldots, j_{r-1}$ is incomparable and intersecting).

Fig. 1. Lexicographic enumeration of $Q(r,n)$ for $r = 1, \dots m$

begin
  read(m,n); $r := 1; j_r := 2$;
  repeat
    if $j_1, \dots, j_r$ is incomparable and intersecting **then**
      begin
      print out $j_1, \dots, j_r$;
      if $j_r < 2^{n-1} - 1$ **then** extend **else** reduce;
      end
    else cut;
  until $j_1 = 2^{n-1} - 1$
end;
extend≡ **begin if** $r < m$ **then** $\{ j_{r+1} := j_r + 1; r := r + 1 \}$ **else** $j_r := j_r + 1$ **end**;
reduce≡ **begin** $r := r - 1; j_r := j_r + 1$ **end**;
cut≡ **if** $j_r < 2^{n-1} - 1$ **then** $j_r := j_r + 1$ **else** reduce.

We have enumerated all incomparable and intersecting sets for $n = 6, 7$. The computation time needed for $n = 7$ case is 7 minutes by a computer FACOM M780 (executing about 30 MIPS).

The data in Table 1 is obtained by this algorithm (up to $n = 5$ one can calculate by hands). We give the numbers of n-variable clique functions $|N(n)|$ in Table 2.

5. Concluding discussions

We have shown that the formula given in Theorem 2 reduces the calculation of $|N(n)|$ to the enumeration of the numbers $k(0, n), \dots, k(m, n)$ which are now feasible for small numbers of n (up to ≤ 7).

Define a graph G (called *intersection graph*) by setting: the set of vertices $:= E^n \setminus 0 \cdots 0$ and the set of edges := two vertices $\boldsymbol{a}$ and $\boldsymbol{b}$ are connected if and only if $\boldsymbol{a}$ and $\boldsymbol{b}$ are intersecting. We show G for $n = 3$ in Fig. 2. Then the number $|N(n)|$ equals the number of cliques of G, that is the sum of the numbers of 0-cliques, 1-cliques, 2-cliques,..., and m-cliques (the size of a maximal clique in the intersection graph G is denoted by m as in Theorem 1).

For the dual "clique" functions

$$N' = \{f \mid \text{if } f(x_1, ..., x_n) = f(y_1, ..., y_n) = 0 \text{ then } x_i = y_i = 0 \text{ for some } i\},$$

we have the same result because of duality.

We may investigate relations of the set of clique functions and the set of monotone functions. This may help us to understand both problems which remain now as difficult enumeration problems.

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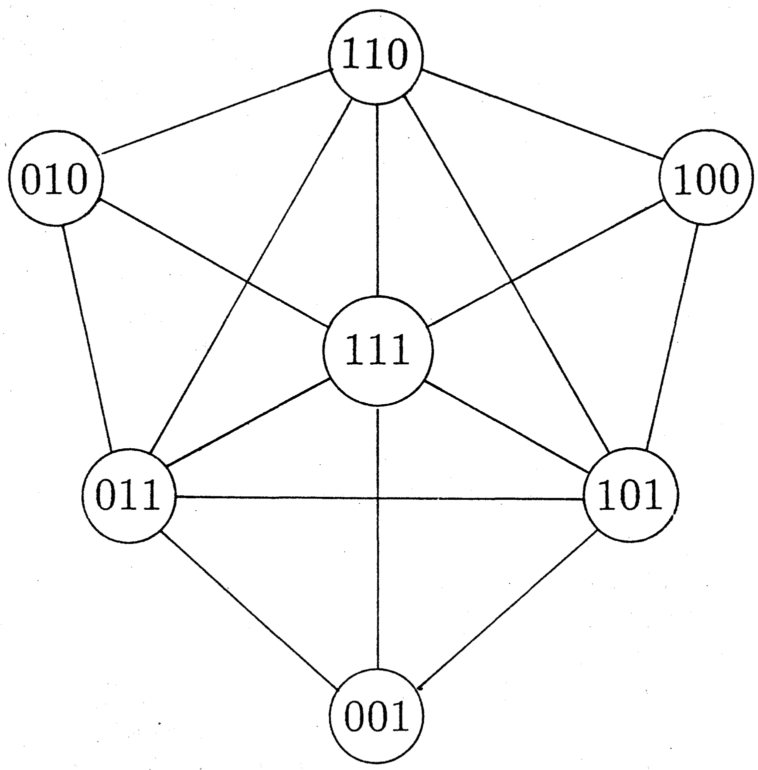

0-clique	(empty set)	1
1-cliques	(vertices)	7
2-cliques	(edges)	15
3-cliques	(Δs)	13
4-cliques	(⊠s)	4
$\lvert N(3)\rvert=$		40

Fig. 2. Intersection graph for $n = 3$.

Table 1. Numbers of $k(r,n)$

	$n=1$ $m=0$	$n=2$ $m=1$	$n=3$ $m=1$	$n=4$ $m=3$	$n=5$ $m=4$	$n=6$ $m=10$	$n=n=7$ $m=15$
$r\backslash$	$k(r,1)$	$k(r,2)$	$k(r,3)$	$k(r,4)$	$k(r,5)$	$k(r,6)$	$k(r,7)$
0	1	1	1	1	1	1	1
1		1	3	7	15	31	63
2				3	30	195	1,050
3				1	30	605	9,030
4					5	780	41,545
5						543	118,629
6						300	233,821
7						135	329,205
8						45	327,915
9						10	224,280
10						1	100,716
11							29,337
12							5,950
13							910
14							105
15							7

Table 2. Numbers of clique functions $N(n)$

$N(1) =$	2
$N(2) =$	6
$N(3) =$	40
$N(4) =$	1,376
$N(5) =$	1,314,816
$N(6) =$	912,818,962,432
$N(7) =$	291,201,248,266,450,683,035,648

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